Thick & Thin cylinders

Objectives:
- Introduction
- Thin wall pressure vessel
- Stress in the Thin cylinders
- Cylindrical pressure vessels
- Thick cylinders
- Stress in the Thick cylinders
- Development of Lame’s equation (1833)
- Error in the Thin Cylinder Formula
- Problem sheet

Introduction

Thin wall pressure vessel

Stress in the Thin cylinders

Cylindrical pressure vessels

Thick cylinders

Stress in the Thick cylinders

Development of Lame’s equation (1833)

Error in the Thin Cylinder Formula

Problem sheet

A pressure vessel is used for storing liquid or under pressure. A pipe line through which pressurized fluid flows is treated as pressure vessel. Normally pressure vessels are of cylindrical or spherical shape.

There are several examples of pressure vessels which are used for engineering purpose. They include boilers, gas storage tanks, metal tires & pipelines.

Thin cylinders

- If the wall thickness of the cylinder is less than 1/20th of the internal diameter ‘di’, the variation of the tangential stresses through the wall thickness is small & the radial stresses may be neglected. The solution can be then treated as statically determinate & the vessel is said to be thin pressure vessel. Thus a thin pressure vessel is one whose thickness to inner radius ratio is not greater than 1/10.

Figure

The stress produced in the longitudinal direction is σL and in the circumferential direction is σC. These are called the longitudinal and circumferential stresses respectively. The latter is also called the hoop stress.

Consider the forces trying to split the cylinder about a circumference (fig.). So long as the wall thickness is small compared to the diameter then the force trying to split it due to the pressure is

\[ F = pA = \frac{\pi D^2}{4} \]

(1.1)

So long as the material holds then the force is balanced by the stress in the wall. The force due to the stress is
Thick Cylinders

• The problem of determination of stresses in a thick cylinders was first attempted more than 160 years ago by a French mathematician Lame in 1833. His solution very logically assumed that a thick cylinder to consist of series of thin cylinders such that each exerts pressure on the other.

This will essentially focus attention on three stress components at any point these stress components are:

• 1) Stress along the circumferential direction, called hoop or tangential stress.

• 2) Radial stress which is stress similar to the pressure on free internal or external surface. (This stress will also vary in the radial direction & not with ‘Θ’ as in tangential stress case.)

• 3) Longitudinal stress in the direction the axis of the cylinder. This stress is perpendicular to the plane of the paper. So the longitudinal stress will remain same/constant for any section of the thick cylinder.

This will be associated with the assumption that any section of thick cylinder will remain plane before & after the application of pressure.

• This assumption will mean that the strain along the axis or length remain constant.

• Thick cylinders also have the external pressure, not only the internal pressure.

\[ F = \sigma_l \times \text{area of the metal} = \sigma_l \pi Dl \] (1.2)

Equating 1.1 and 1.2 we have

\[ \sigma_l = \frac{pD}{4l} \] (1.3)

Now consider the forces trying to split the cylinder along a length. The force due to the pressure is

\[ F = pA = pLD \] (1.4)

So long as the material holds this is balanced by the stress in the material. The force due to the stress is

\[ F = \sigma_c \times \text{area of the metal} = \sigma_c 2Ll \] (1.5)

Equating 1.4 and 1.5 we have

\[ \sigma_c = \frac{pD}{2l} \] (1.6)

It follows that for a given pressure the circumferential stress is twice the longitudinal stress.
Lame’s Theory

Consider a small section of the wall.

\[ \sigma_L - \text{Longitudinal stress} \]
\[ \sigma_R - \text{Radial stress} \]
\[ \sigma_C - \text{Circumferential stress} \]

Figure 6

We have 3 stresses in mutually perpendicular directions, the corresponding strains are

\[ \varepsilon_L = \frac{1}{E} [\sigma_L - \nu (\sigma_R + \sigma_C)] \]
\[ \varepsilon_C = \frac{1}{E} [\sigma_C - \nu (\sigma_L + \sigma_R)] \]
\[ \varepsilon_R = \frac{1}{E} [\sigma_R - \nu (\sigma_C + \sigma_L)] \]

For small angles:

\[ \left( \sigma_r + \delta \sigma_r \right) (r + dr) d\theta \times 1 - \sigma_r \times rd\theta \times 1 = 2\sigma_H \times dr \times 1 \times \sin \frac{d\theta}{2} \]

Therefore, neglecting second-order small quantities,

\[ r d\sigma_r + \sigma_r dr = \sigma_H dr \]

\[ \sigma_r + \frac{d\sigma_r}{dr} = \sigma_H \]

\[ \sigma_H - \sigma_r = \frac{d\sigma_r}{dr} \quad \text{(A)} \]

Assuming now that plane sections remain plane, i.e., the longitudinal strain is constant across the wall of the cylinder,

\[ \varepsilon_L = \frac{1}{E} [\sigma_L - \nu \sigma_r - \nu \sigma_H] \]

\[ = \frac{1}{E} [\sigma_L - \nu (\sigma_r + \sigma_H)] = \text{constant} \]

• It is also assumed that the longitudinal stress is constant across the cylinder walls at points remote from the ends.

\[ \sigma_r + \sigma_H = \text{constant} = 2A \text{ (say)} \quad \text{----- (B)} \]
The above equations yield the radial and hoop stresses at any radius \( r \) in terms of constants \( A \) and \( B \). For any pressure condition there will always be two known conditions of stress.

Consider now the thick cylinder shown in Fig. subjected to an internal pressure \( P \), the external pressure being zero.

The two known conditions of stress which enable the Lamé constants \( A \) and \( B \) to be determined are:

At \( r = R_1 \) \( \sigma_r = -P \) and at \( r = R_2 \) \( \sigma_r = 0 \)

Substituting the Lamé’s Equations

\[
-P = A - \frac{B}{R_1^2}
\]

\[
0 = A - \frac{B}{R_2^2}
\]

i.e.

\[
A = \frac{PR_1^2}{(R_2^2 - R_1^2)} \quad \text{and} \quad B = \frac{PR_1^2R_2^2}{(R_2^2 - R_1^2)}
\]

\[
\sigma_r = A - \frac{B}{r^2}
\]

\[
= \frac{PR_1^2}{(R_2^2 - R_1^2)} \left[ 1 - \frac{R_2^2}{r^2} \right]
\]

\[
= \frac{PR_1^2}{(R_2^2 - R_1^2)} \left[ \frac{r^2 - R_2^2}{r^2} \right] = -P \left[ \frac{(R_2/r)^2 - 1}{k^2 - 1} \right]
\]

where \( k \) is the diameter ratio \( D_2/D_1 = R_2/R_1 \)

and hoop stress \( \sigma_H = \frac{PR_1^2}{(R_2^2 - R_1^2)} \left[ 1 + \frac{R_2^2}{r^2} \right] \)
As \( r = R_1 \)
\[
\sigma_h = p_i \left[ \frac{K^2 + 1}{K^2 - 1} \right]
\]
similarly,
\[
\sigma_r = -p_i
\]

To find the Max. error in the thin cylinder formula, put the Max. i.e. the limiting value for \( t = \frac{R_1}{10} \) or \( \frac{R_1}{t} = 10 \)
\[
\Rightarrow \sigma_h = p_i \left\{ \frac{2(10)^2 + 2(10) + 1}{2(10) + 1} \right\} = 10.52 p_i
\]
And from the Thin Cylinder Formula:
\[
\sigma_h = p_i \frac{R_1}{t} = 10p_i \text{ As } \frac{R_1}{t} = 10p_i
\]
\[
\%age \text{ Error} = 10.52p_i - 10p_i / 10p_i = 5.2 \%
\]