Thick & Thin cylinders

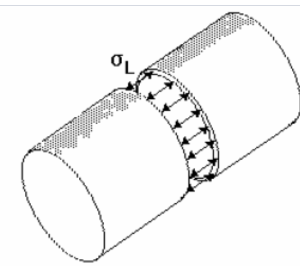
Objectives:-

- Introduction
- Thin wall pressure vessel
- Stress in the Thin cylinders
- Cylindrical pressure vessels
- Thick cylinders
- Stress in the Thick cylinders
- Development of Lame's equation (1833)
- Error in the Thin Cylinder Formula
- Problem sheet

- A pressure vessel is used for storing liquid or under pressure. A pipe line through which pressurized fluid flows is treated as pressure vessel. Normally pressure vessels are of cylindrical or spherical shape.
- There are several examples of pressure vessels which are used for engineering purpose. They include boilers, gas storage tanks, metal tires & pipelines

Thin cylinders

- If the wall thickness of the cylinder is less than 1/20th of the internal diameter 'di',the variation of the tangential stresses through the wall thickness is small & the radial stresses may be neglected. The solution can be then treated as statically determinate & the vessel is said to be thin pressure vessel. Thus a thin pressure vessel is one whose thickness to inner radius ratio is not greater than 1/10.



Figure

The stress produced in the longitudinal direction is σ_L and in the circumferential direction is σ_c. These are called the longitudinal and circumferential stresses respectively. The latter is also called the hoop stress.

Consider the forces trying to split the cylinder about a circumference (fig). So long as the wall thickness is small compared to the diameter then the force trying to split it due to the pressure is

$$F = pA = p\frac{\pi D^2}{4} \text{.........................}(1.1)$$

So long as the material holds then the force is balanced by the stress in the wall. The force due to the stress is

$F = \sigma_L$ multiplied by the area of the metal $= \sigma_L \pi D t$ ....(1.2)

Equating 1.1 and 1.2 we have

$$\sigma_L = \frac{pD}{4t} \quad \text{.................................(1.3)}$$

Now consider the forces trying to split the cylinder along a length.
The force due to the pressure is

$$F = pA = pLD \quad \text{............................(1.4)}$$

So long as the material holds this is balanced by the stress in the material. The force due to the stress is

$F = \sigma_C$ multiplied y the area of the metal $= \sigma_C 2Lt$ ....(1.5)

Equating 1.4 and 1.5 we have

$$\sigma_C = \frac{pD}{2t} \quad \text{........................(1.6)}$$

It follows that for a given pressure the circumferential stress is twice the longitudinal stress.

Thick Cylinders

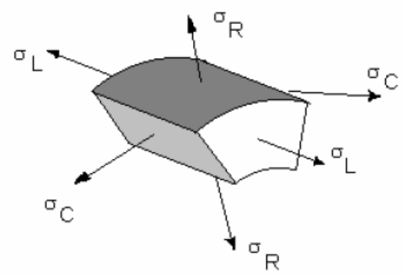
- The problem of determination of stresses in a thick cylinders was first attempted more than 160 years ago by a French mathematician Lame in 1833. His solution very logically assumed that a thick cylinder to consist of series of thin cylinders such that each exerts pressure on the other.

- This will essentially focus attention on three stress components at any point these stress components are:
- 1) Stress along the circumferential direction, called hoop or tangential stress.
- 2) Radial stress which is stress similar to the pressure on free internal or external surface. (This stress will also vary in the radial direction & not with 'ϴ' as in tangential stress case.)
- 3) Longitudinal stress in the direction the axis of the cylinder. This stress is perpendicular to the plane of the paper. So the longitudinal stress will remain same /constant for any section of the thick cylinder.

- This will be associated with the assumption that any section of thick cylinder will remain plane before & after the application of pressure.
- This assumption will mean that the strain along the axis or length remain constant.
- Thick cylinders also have the external pressure, not only the internal pressure.

Lame's Theory

Consider a small section of the wall.



σ_L = Longitudinal stress

σ_R = Radial stress

σ_C = Circumferential stress

Figure 6

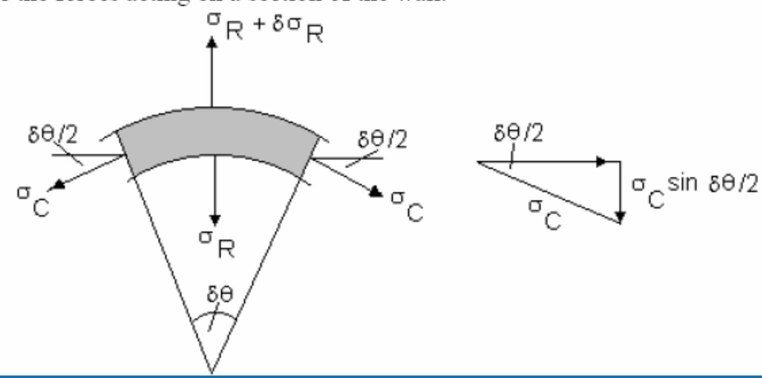
We have 3 stresses in mutually perpendicular directions, the corresponding strains are

$$\varepsilon_l = \frac{1}{E}\{\sigma_L - \nu(\sigma_R + \sigma_C)\}$$

$$\varepsilon_C = \frac{1}{E}\{\sigma_C - \nu(\sigma_L + \sigma_R)\}$$

$$\varepsilon_R = \frac{1}{E}\{\sigma_R - \nu(\sigma_C + \sigma_L)\}$$

consider the forces acting on a section of the wall.



$$(\sigma_r + d\sigma_r)(r + dr)\,d\theta \times 1 - \sigma_r \times r\,d\theta \times 1 = 2\sigma_H \times dr \times 1 \times \sin\frac{d\theta}{2}$$

For small angles: $\sin\frac{d\theta}{2} \simeq \frac{d\theta}{2}$

Therefore, neglecting second-order small quantities,

$$r\,d\sigma_r + \sigma_r\,dr = \sigma_H\,dr$$

$$\sigma_r + r\frac{d\sigma_r}{dr} = \sigma_H$$

$$\sigma_H - \sigma_r = r\frac{d\sigma_r}{dr} \qquad \text{——— (A)}$$

Assuming now that plane sections remain plane, Le. the longitudinal strain is constant across the wall of the cylinder,

$$\varepsilon_L = \frac{1}{E}[\sigma_L - \nu\sigma_r - \nu\sigma_H]$$

$$= \frac{1}{E}[\sigma_L - \nu(\sigma_r + \sigma_H)] = \text{constant}$$

- It is also assumed that the longitudinal stress is constant across the cylinder walls at points remote from the ends.

$$\sigma_r + \sigma_H = \text{constant} = 2A \text{ (say)}$$

------ **(B)**

Substituting in (A) for σ_H,

$$2A - \sigma_r - \sigma_r = r\frac{d\sigma_r}{dr}$$

Multiplying through by r and rearranging,

$$2\sigma_r r + r^2\frac{d\sigma_r}{dr} - 2Ar = 0$$

i.e.

$$\frac{d}{dr}(\sigma_r r^2 - Ar^2) = 0$$

Therefore, integrating,

$$\sigma_r r^2 - Ar^2 = \text{constant} = -B \text{ (say)}$$

$\therefore$

$$\sigma_r = A - \frac{B}{r^2}$$

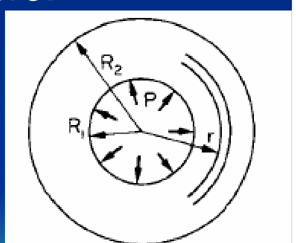
and from eqn. (B)

$$\sigma_H = A + \frac{B}{r^2}$$

- The above equations yield the radial and hoop stresses at any radius ***r*** in terms of constants *A* and B. For any pressure condition there will always be two known conditions of stress

Thick cylinder - internal pressure only

- Consider now the thick cylinder shown in Fig. subjected to an internal pressure P, *the external pressure being zero.*
- The two known conditions of stress which enable the Lame constants **A** and ***B*** to be determined are:



$$\text{At } r = R_1 \quad \sigma_r = -P \quad \text{and} \quad \text{at } r = R_2 \quad \sigma_r = 0$$

Substituting the Lame's Equations

$$-P = A - \frac{B}{R_1^2}$$

$$0 = A - \frac{B}{R_2^2}$$

i.e.

$$A = \frac{PR_1^2}{(R_2^2 - R_1^2)} \quad \text{and} \quad B = \frac{PR_1^2 R_2^2}{(R_2^2 - R_1^2)}$$

$\therefore$

$$\text{radial stress } \sigma_r = A - \frac{B}{r^2}$$

$$= \frac{PR_1^2}{(R_2^2 - R_1^2)}\left[1 - \frac{R_2^2}{r^2}\right]$$

$$= \frac{PR_1^2}{(R_2^2 - R_1^2)}\left[\frac{r^2 - R_2^2}{r^2}\right] = -P\left[\frac{(R_2/r)^2 - 1}{k^2 - 1}\right]$$

where k is the diameter ratio $D_2/D_1 = R_2/R_1$

and

$$\text{hoop stress } \sigma_H = \frac{PR_1^2}{(R_2^2 - R_1^2)}\left[1 + \frac{R_2^2}{r^2}\right]$$

$$= \frac{PR_1^2}{(R_2^2 - R_1^2)}\left[\frac{r^2 + R_2^2}{r^2}\right] = P\left[\frac{(R_2/r)^2 + 1}{k^2 - 1}\right]$$ ---

$$= \frac{PR_1^2}{(R_2^2 - R_1^2)}\left[\frac{R_1^2 + R_2^2}{R_1^2}\right]$$

------- (D)

As r = R1

$\sigma_h = p_i [K^2 + 1] / [K^2 - 1]$

similarly,

$\sigma_r = -p_i$

Errors in the Thin Cylinder Formula:

At inner Radius when $r = R_1$ & P = pi the Equation (c) becomes

- As t = R2 – R1
- ➔ R2 = R1 + t

Substitute the values in Equation (D)

➔ $\sigma_h = p_i \{ R_1^2 + (R_1+t)^2 \} / \{ (R_1+t)^2 - R_1^2 \}$ \

➔ $= p_i \{2(R_1/t)^2 + 2(R_1/t) + 1\} / 2(R_1/t) + 1$

For the Thin cylinders $t < R_1 / 10$

- To find the Max. error in the thin cylinder formula, put the Max. i.e. the limiting value for $t = R_1 / 10$ or $R_1 / t = 10$

  ➔ $\sigma_h = p_i \{2(10)^2 + 2(10) + 1\} / 2(10) + 1$

  $= 10.52\ p_i$

And from the Thin Cylinder Formula:

$\sigma_h = p_i R_1 / t = 10p_i$ As $R_1 / t = 10p_i$

%age Error = $10.52p_i - 10p_i / 10p_i = 5.2$ %