Total Strain Energy Theory (Beltrami-Haigh)

It predicts the failure of a specimen subjected to any combination of loads when the strain energy per unit volume of any portion of the stressed member reaches the failure value of strain energy per unit volume as determined from an axial or compression test of the same material.

- The total strain energy per unit volume is given by the sum of the energy component due to three principal stresses and strains.
- For a 3 - dimensional state of stress system the total strain energy U_t per unit volume in equal to the total work done by the system and given by the equation

- $U_t = ½\ \sigma_1\varepsilon_1 + ½\ \sigma_2\varepsilon_2 + ½\ \sigma_3\varepsilon_3$
- $\varepsilon_1 = \sigma_1 / E - v\sigma_2 / E - v\sigma_3 / E$
- $\varepsilon_2 = \sigma_2 / E - v\sigma_1 / E - v\sigma_3 / E$
- $\varepsilon_3 = \sigma_3 / E - v\sigma_2 / E - v\sigma_1 / E$
- =>
- **$U_t = 1/2E\ [\ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\ ]$**

- For 2-D case;
- $U_t = ½ E [\sigma_1^2 + \sigma_2^2 -2v(\sigma_1\sigma_2)]$

According to this theory the materials fails when this energy equals to the limiting Energy

$U_L = ½ \sigma_L\varepsilon_L = ½ \sigma_L \sigma_L /E = ½ \sigma_L^2 /E$

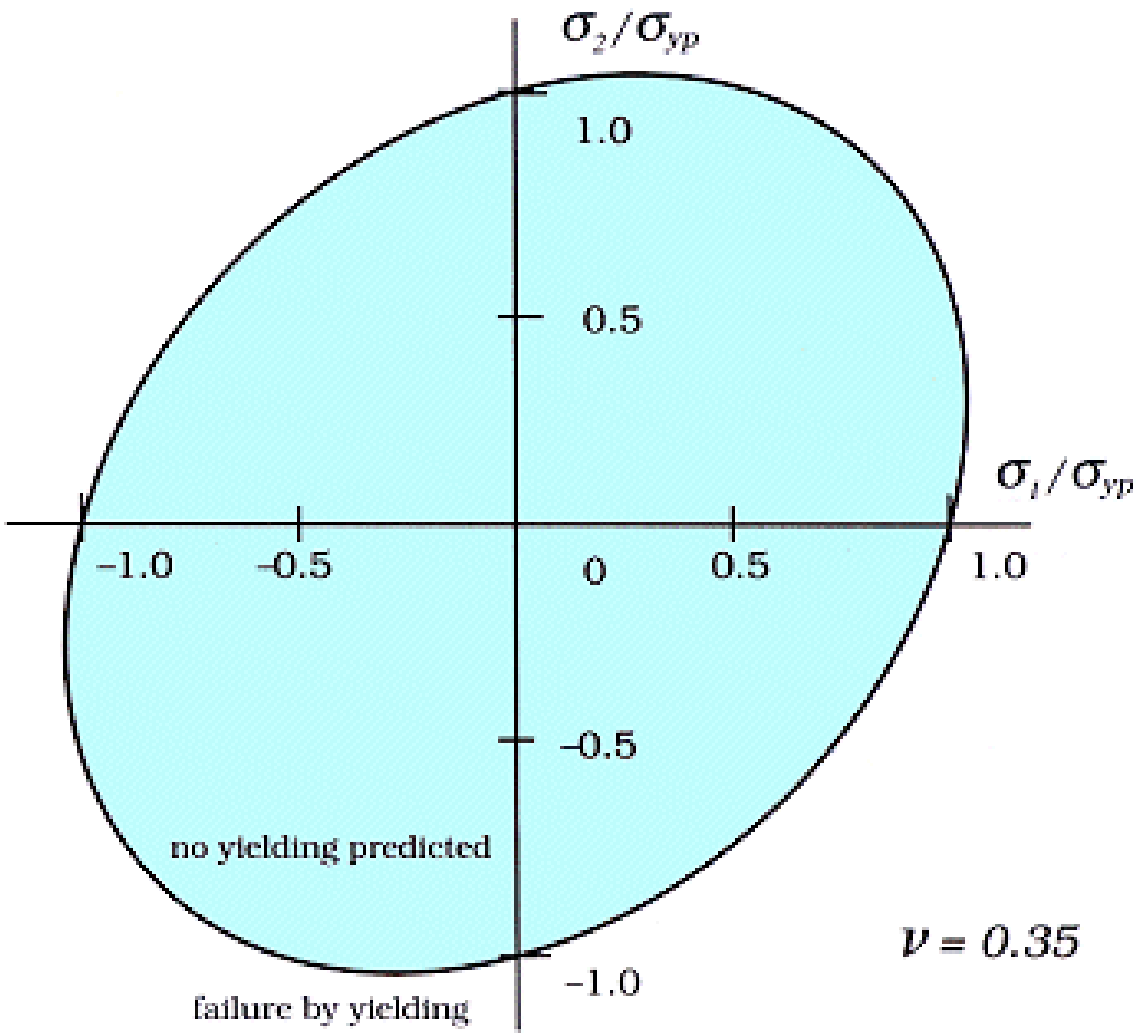
i.e.

$$\tfrac{1}{2} E \left[\sigma_1^2 + \sigma_2^2 - 2v(\sigma_1\sigma_2)\right] = \tfrac{1}{2} \sigma_L^2 / E$$

$$\Rightarrow (\sigma_1 / \sigma_L)^2 + (\sigma_2 / \sigma_L)^2 - 2v(\sigma_1 / \sigma_L)(\sigma_2 / \sigma_L) = 1$$

This is an equation of ellipse with axes at 45^0

2-D Yield Locus



Maximum Distortion Energy Theory (Huber-Henky-von Mises)

- It predicts the failure of a specimen subjected to any combination of loads when the strain energy per unit volume due to shear of any portion of the stressed member reaches the failure value of strain energy per unit volume due to shear as determined from an axial or compression test of the same material.

- The total strain energy per unit volume is given by the sum of the energy component due to three principal stresses and strains.
- $U_t = ½\, \sigma_1\varepsilon_1 + ½\, \sigma_2\varepsilon_2 + ½\, \sigma_3\varepsilon_3$
- $\varepsilon_1 = \sigma_1 / E - v\sigma_2 / E - v\sigma_3 / E$
- $\varepsilon_2 = \sigma_2 / E - v\sigma_1 / E - v\sigma_3 / E$
- $\varepsilon_3 = \sigma_3 / E - v\sigma_2 / E - v\sigma_1 / E$

=>

- $\mathbf{U_t = 1/2E\, [\, \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\,]}$

---------- (1)

- Here the total strain energy can be considered as the sum of two parts, one part representing the energy needed to cause a volume change of the element with no change in shape & the other part representing the energy needed to distort the element.
- i.e.
- $\mathbf{U_t = U_v + U_s}$
- Or $\mathbf{U_s = U_t - U_v}$

- $U_v = ½\ \sigma_v \varepsilon_v$

$$= ½\ \sigma_v \{ 3\ \sigma_v\ (1\text{-}2v) / E \}$$

$$= 3(1\text{-}2v)\ \sigma_v^2 / 2E$$

$$= 3(1\text{-}2v)\ \{(\sigma_1 + \sigma_2 + \sigma_3 /3)^2\} / 2E$$

$$= 1\text{-}2v\ (\sigma_1 + \sigma_2 + \sigma_3)^2 /6$$

$$\mathbf{U_s = 1+ v / 6E\ [\ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

---------- (2)

- So for the case of Maximum shear/ distortion energy theory, the failure occurs when the quantity Us reaches the value in elastic limit. As for limiting value

$\sigma_2 = \sigma_3 = 0$ & $\sigma_1 = \sigma_L$

$U_L = 1 + v / 6E [\sigma_L^2 + \sigma_L^2]$

$= 1 + v / 3E (\sigma_L)^2$ -------- (3)

- At failure:

$$U_s = U_L$$

So Equation # (2) & (3) ➔

$$1 + v / 6E\,[\,(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 1 + v / 3E\,(\sigma_L)^2$$

For 2-D Case

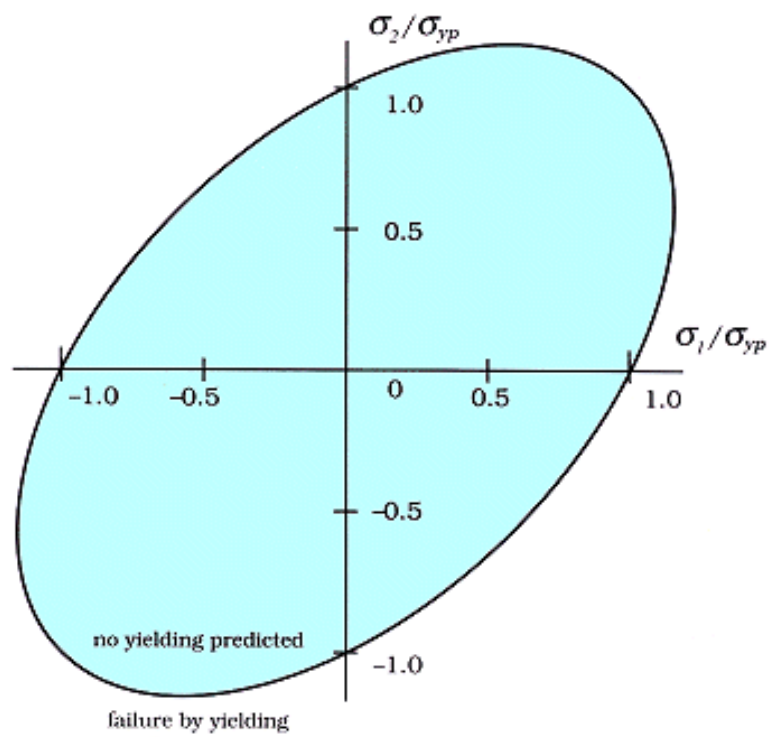
$$1 + v / 3E\,(\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2) = 1 + v / 3E\,(\sigma_L)^2$$

$$(\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2) = (\sigma_L)^2$$

➔ $(\sigma_1 / \sigma_L)^2 + (\sigma_2 / \sigma_L)^2 - (\sigma_1 / \sigma_L)(\sigma_2 / \sigma_L) = 1$

- This is an equation of ellipse & same as previous but with bigger locus than the previous.

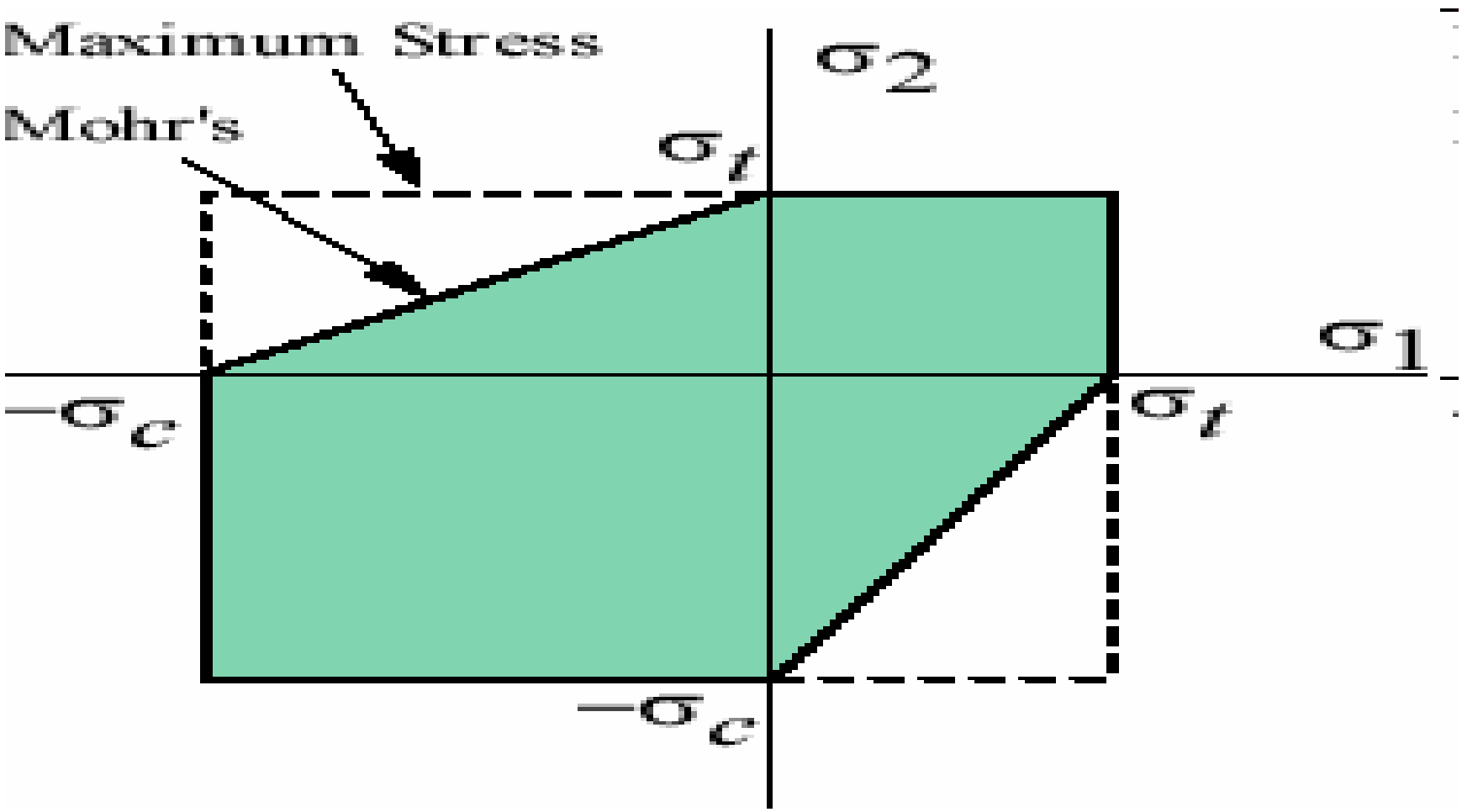
- 2-D yield locus



Mohr's failure criterion

- Some materials such as rocks, concrete, cast iron has much greater strength in compression then in tension.
- Mohr's proposed that, in 1st and 3rd quadrant of the failure Maximum Principal Stress Theory was appropriate based on the ultimate strength of the material in tension or compression respectively.
- In 2nd & 4th quadrant the Maximum Shear Stress Theory should be applied.

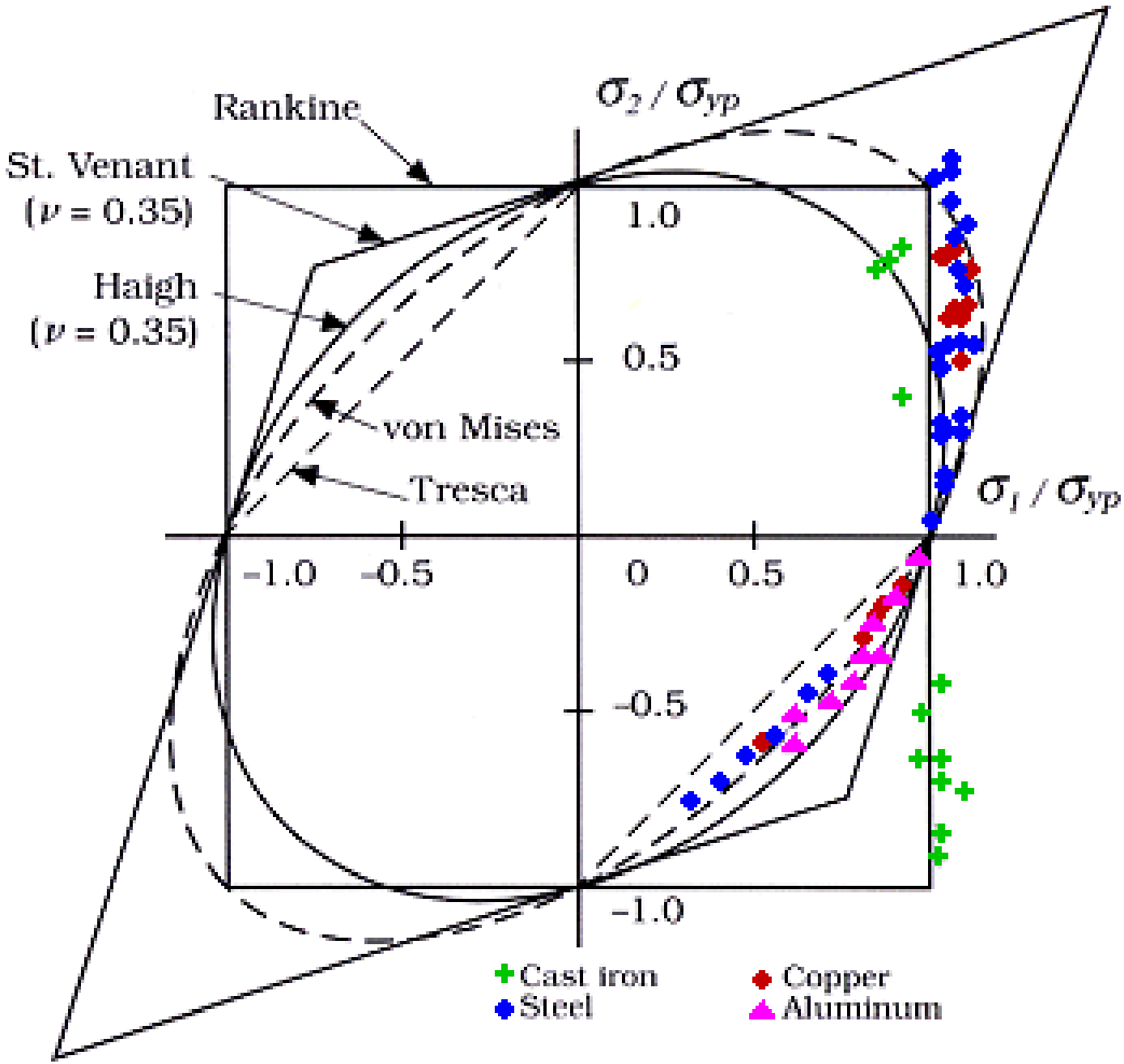
Yield Locus


Maximum Stress
Mohr's
σ_2
σ_t
σ_1
$-\sigma_c$
σ_t
$-\sigma_c$

Conclusion / Summary

- Materials does not fail under hydrostatic stress system i.e $\sigma 1 = \sigma 2 = \sigma 3$
- None of the theories agrees with the test perform for all types of materials and combinations of loads.
- There is a good agreement between the maximum distortion energy theory and experimental result for ductile materials.
- The max. principal stress theory appears to be the best for brittle materials

- Max. shear stress or max. strain energy theories give the good approximation for ductile materials but the max. shear stress criterion is somewhat more conservative.
- The max. strain theory should not be used in general as it only gives the reliable results in particular cases.
- If the brittle material has a stress strain diagram, that is different in tension and compression, then the MOHR'S Failure Criterion may be used to predict the failure.


Rankine
St. Venant
(ν = 0.35)
Haigh
(ν = 0.35)
von Mises
Tresca
σ2 / σyp
σ1 / σyp
1.0
0.5
0
-0.5
-1.0
-1.0
-0.5
0.5
1.0
Cast iron
Steel
Copper
Aluminum