Chapter 7

Centroids, Moments of Inertia, and Products of Inertia of Plane Areas

FIRST MOMENT OF AN ELEMENT OF AREA

The first moment of an element of area about any axis in the plane of the area is given by the product of the area of the element and the perpendicular distance between the element and the axis. For example, in Fig. 7-1 the first moment dQ_x of the element da about the x-axis is given by

$$dQ_x = y\,da$$

About the y-axis the first moment is

$$dQ_y = x\,da$$

For applications, see Problems 7.2 and 7.12.

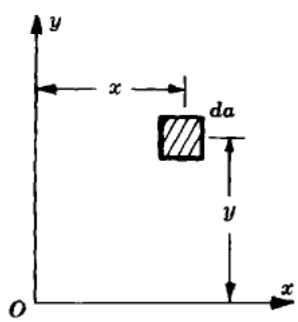


Fig. 7-1

FIRST MOMENT OF A FINITE AREA

The first moment of a finite area about any axis in the plane of the area is given by the summation of the first moments about that same axis of all the elements of area contained in the finite area. This is frequently evaluated by means of an integral. If the first moment of the finite area is denoted by Q_x, then

$$Q_x = \int dQ_x$$

For applications, see Problems 7.1 and 7.3.

CENTROID OF AN AREA

The centroid of an area is defined by the equations

$$\bar{x} = \frac{\int x\,da}{A} = \frac{Q_y}{A} \qquad \bar{y} = \frac{\int y\,da}{A} = \frac{Q_x}{A}$$

where A denotes the area. For a plane area composed of N subareas A_i each of whose centroidal coordinates $\bar{x}_i$ and $\bar{y}_i$ are known, the integral is replaced by a summation

$$\bar{x} = \frac{\sum_{i=1}^{N} \bar{x}_i A_i}{\sum_{i=1}^{N} A_i} \tag{7.1}$$

$$\bar{y} = \frac{\sum_{i=1}^{N} \bar{y}_i A_i}{\sum_{i=1}^{N} A_i} \tag{7.2}$$

For applications see Problems 7.2, 7.3, and 7.12.

The centroid of an area is the point at which the area might be considered to be concentrated and still leave unchanged the first moment of the area about any axis. For example, a thin metal plate will balance in a horizontal plane if it is supported at a point directly under its center of gravity.

The centroids of a few areas are obvious. In a symmetrical figure such as a circle or square, the centroid coincides with the geometric center of the figure.

It is common practice to denote a centroid distance by a bar over the coordinate distance. Thus $\bar{x}$ indicates the x-coordinate of the centroid.

SECOND MOMENT, OR MOMENT OF INERTIA, OF AN ELEMENT OF AREA

The second moment, or moment of inertia, of an element of area about any axis in the plane of the area is given by the product of the area of the element and the square of the perpendicular distance between the element and the axis. In Fig. 7-1, the moment of inertia dI_x of the element about the x-axis is

$$dI_x = y^2\,da$$

About the y-axis the moment of inertia is

$$dI_y = x^2\,da$$

SECOND MOMENT, OR MOMENT OF INERTIA, OF A FINITE AREA

The second moment, or moment of inertia, of a finite area about any axis in the plane of the area is given by the summation of the moments of inertia about that same axis of all of the elements of area contained in the finite area. This, too, is frequently found by means of an integral. If the moment of inertia of the finite area about the x-axis is denoted by I_x, then we have

$$I_x = \int dI_x = \int y^2\,da \tag{7.3}$$

$$I_y = \int dI_y = \int x^2\,da \tag{7.4}$$

For a plane area composed of N subareas A_i each of whose moment of inertia is known about the x- and y-axes, the integral is replaced by a summation

$$I_x = \sum_{i=1}^{N} (I_x)_i \qquad I_y = \sum_{i=1}^{N} (I_y)_i$$

For applications, see Problems 7.4, 7.6, 7.7, 7.8, 7.9, and 7.10.

UNITS

The units of moment of inertia are the fourth power of a length, in^4 or m^4.

PARALLEL-AXIS THEOREM FOR MOMENT OF INERTIA OF A FINITE AREA

The parallel-axis theorem for moment of inertia of a finite area states that the moment of inertia of an area about any axis is equal to the moment of inertia about a parallel axis through the centroid of the area plus the product of the area and the square of the perpendicular distance between the two axes. For the area shown in Fig. 7-2, the axes x_G and y_G pass through the centroid of the plane area. The x- and y-axes are parallel axes located at distances x_1 and y_1 from the centroidal axes. Let A denote the area of the figure, I_{x_G} and I_{y_G} the moments of inertia about the axes through the centroid, and I_x and I_y the moments of inertia about the x- and y-axes. Then we have

$$I_x = I_{x_G} + A(y_1)^2 \tag{7.5}$$

$$I_y = I_{y_G} + A(x_1)^2 \tag{7.6}$$

This relation is derived in Problem 7.5. For applications, see Problems 7.6, 7.8, 7.11, and 7.12.

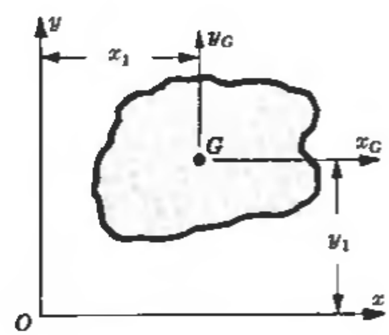


Fig. 7-2

RADIUS OF GYRATION

If the moment of inertia of an area A about the x-axis is denoted by I_x, then the radius of gyration r_x is defined by

$$r_x = \sqrt{\frac{I_x}{A}} \tag{7.7}$$

Similarly, the radius of gyration with respect to the y-axis is given by

$$r_y = \sqrt{\frac{I_y}{A}} \tag{7.8}$$

Since I is in units of length to the fourth power, and A is in units of length to the second power, then the radius of gyration has the units of length, say in or m. It is frequently useful for comparative purposes but has no physical significance. See Problems 7.10 and 7.11.

PRODUCT OF INERTIA OF AN ELEMENT OF AREA

The product of inertia of an element of area with respect to the x- and y-axes in the plane of the area is given by

$$dI_{xy} = xy\,da$$

where x and y are coordinates of the elemental area as shown in Fig. 7-1.

PRODUCT OF INERTIA OF A FINITE AREA

The product of inertia of a finite area with respect to the x- and y-axes in the plane of the area is given by the summation of the products of inertia about those same axes of all elements of area contained within the finite area. Thus

$$I_{xy} = \int xy\,da \tag{7.9}$$

From this, it is evident that I_{xy} may be positive, negative, or zero. For a plane area composed of N subareas A_i each of whose product of inertia is known with respect to specified x- and y-axes, the integral is replaced by the summation

$$I_{xy} = \sum_{i=1}^{N} (I_{xy})_i \tag{7.10}$$

For applications see Problems 7.13 and 7.15.

PARALLEL-AXIS THEOREM FOR PRODUCT OF INERTIA OF A FINITE AREA

The parallel-axis theorem for product of inertia of a finite area states that the product of inertia of an area with respect to the x- and y-axes is equal to the product of inertia about a set of parallel axes passing through the centroid of the area plus the product of the area and the two perpendicular distances from the centroid to the x- and y-axes. For the area shown in Fig. 7.2, the axes x_G and y_G pass through the centroid of the plane area. The x- and y-axes are parallel axes located at distances x_1 and y_1 from the centroidal axes. Let A represent the area of the figure and $I_{x_G y_G}$ be the product of inertia about the axes through the centroid. Then we have

$$I_{xy} = I_{x_G y_G} + A x_1 y_1 \tag{7.11}$$

This relation is derived in Problem 7.14. For applications see Problems 7.15 and 7.16.

PRINCIPAL MOMENTS OF INERTIA

At any point in the plane of an area there exist two perpendicular axes about which the moments of inertia of the area are maximum and minimum for that point. These maximum and minimum values of moment of inertia are termed *principal moments of inertia* and are given by

$$(I_{x_1})_{\max} = \left(\frac{I_x + I_y}{2}\right) + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2} \tag{7.12}$$

$$(I_{x_1})_{\min} = \left(\frac{I_x + I_y}{2}\right) - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2} \tag{7.13}$$

These expressions are derived in Problem 7.17. For application, see Problem 7.18.

PRINCIPAL AXES

The pair of perpendicular axes through a selected point about which the moments of inertia of a plane area are maximum and minimum are termed *principal axes*. For application, see Problem 7.16.

The product of inertia vanishes if the axes are principal axes. Also, from the integral defining product of inertia of a finite area, it is evident that if either the x-axis, or the y-axis, or both, are axes of symmetry, the product of inertia vanishes. Thus, axes of symmetry are principal axes.

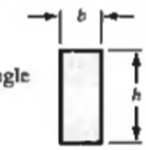
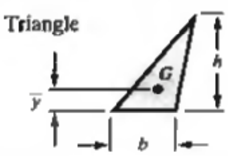
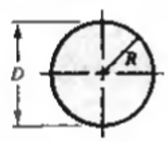
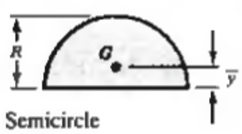
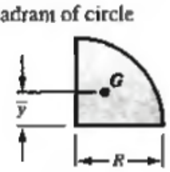
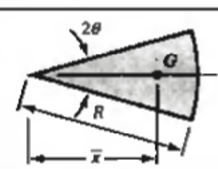
Type of section	Area	Location of centroid
Rectangle  (*a*)	bh	Geometric center
Triangle  (*b*)	$\frac{1}{2}bh$	$\bar{y} = \frac{h}{3}$
Circle  (*c*)	πR^2 or $\frac{\pi}{4}D^2$	Geometric center
 Semicircle (*d*)	$\frac{1}{2}\pi R^2$ or $\frac{\pi}{8}D^2$	$\bar{y} = \frac{4R}{3\pi}$
Quadrant of circle  (*e*)	$\frac{\pi R^2}{4}$	$\bar{y} = \frac{4R}{3\pi}$
 Sector of circle (*f*)	θR^2	$\bar{x} = \frac{2R \sin\theta}{3\theta}$

Fig. 7-3

INFORMATION FROM STATICS

Most texts on statics develop the properties of plane cross-sectional areas shown in Fig. 7-3 that will be needed in the present chapter. Those areas include (*a*) the rectangle, (*b*) the triangle, (*c*) the circle, (*d*) the semicircle, (*e*) the quadrant of a circle, and (*e*) the sector of a circle.

Solved Problems

7.1. The shaded area shown in Fig. 7-4 is bounded by the curves

$$y_1 = \sqrt[3]{x}$$

and

$$y_2 = x^3$$

Determine the y-coordinate of the centroid of this area which ends at (1,1).

We select an element that is horizontal (thus all points in this element have the same "y") and

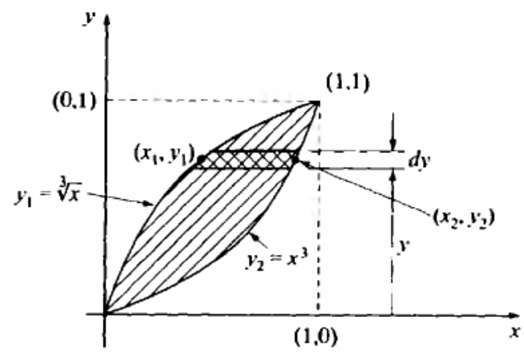


Fig. 7-4

extending from curve y_1 to y_2 as shown in Fig. 7-4. The height of the element is dy. From the definition of the location of the centroid,

$$\bar{y} = \frac{\int y\, da}{A}$$

we can write

$$da = (x_2 - x_1)\,dy$$

in which case we have

$$\bar{y} = \frac{\int_0^1 (x_2 - x_1)(y)(dy)}{\int_0^1 (x_2 - x_1)\,dy}$$

$$= \frac{\int_0^1 (y^{1/3} - y^3)(y)(dy)}{\int_0^1 (y^{1/3} - y^3)\,dy} = \frac{16}{70} = 0.229$$

Although the integrations involved in this problem are simple, for more complex problems one should resort to computers. A number of symbolic operations are available on proprietary software that permit easy and rapid treatments of such computations.

7.2. A circular cross section has a sector having a central angle 2θ removed as shown in Fig. 7-5. Locate the y-coordinate of the centroid of the shaded area.

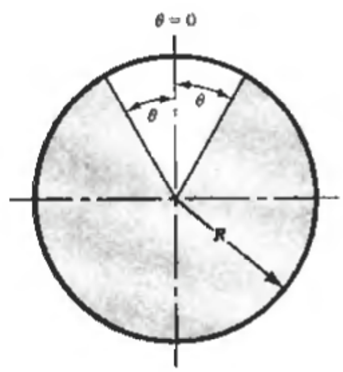


Fig. 7-5

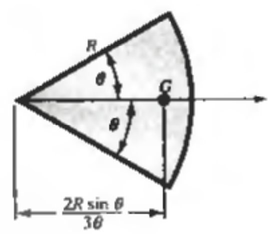


Fig. 7-6

From the summary at the beginning of this chapter, we have for a sector of central angle 2θ the area and centroid given by θR^2 and $2R\sin\theta/3\theta$, respectively (see Fig. 7-6). The area of the entire circle having its centroid at its geometric center is also given in that summary.

By definition the y-coordinate of the centroid of the shaded area in Fig. 7-4 is given by

$$\bar{y} = \frac{\int y\,da}{A} \quad \text{or} \quad \frac{\Sigma y\,da}{A}$$

Here we consider the shaded area to be composed of the three components consisting of the lower semicircle ①, the upper semicircle ②, and the sector that has been removed ③. Thus the net shaded area is represented as shown in Fig. 7-7.

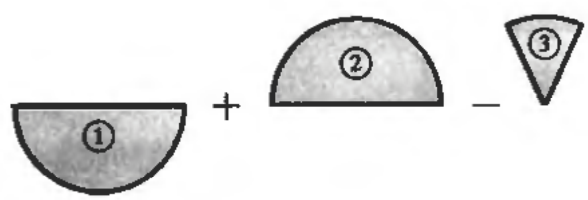


Fig. 7-7

Using these components in the finite summation (*7.1*), we have

① ② ③

$$\bar{y} = \frac{\dfrac{\pi}{2}R^2\left(-\dfrac{4R}{3\pi}\right) + \dfrac{\pi}{2}R^2\left(\dfrac{4R}{3\pi}\right) - \theta R^2\left(\dfrac{2R}{3\theta}\sin\theta\right)}{\pi R^2 - \theta R^2}$$

$$= -\frac{\frac{2}{3}(R\sin\theta)}{(\pi - \theta)}$$

7.3. A thin sheet of metal 600 mm by 1000 mm has its two upper corners folded over along the inclined lines AC and DF as shown in Fig. 7-8. In the regions bounded by the dotted lines, the metal thus becomes doubly thick. Determine the y-coordinate of the centroid of the folded sheet.

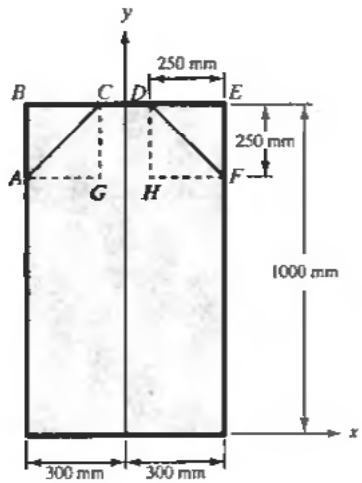


Fig. 7-8

By definition, the y-coordinate of the centroid is

$$\bar{y} = \frac{\int y\,da}{A} \qquad \text{or} \qquad \frac{\Sigma\, y_i A_i}{A}$$

where the numerator in each expression represents the first moment of the area about the x-axis. In the numerical evaluation, the triangles ABC and DEF have been removed but replaced by triangles ACG and DFH accounting for the double thickness. Thus we have

$$\bar{y} = \frac{(600)(1000)(500) - \overbrace{2\{\tfrac{1}{2}(250)(250)[1000 - \tfrac{250}{3}]\}}^{\triangle BCA} + \overbrace{2\{\tfrac{1}{2}(250)(250)[750 + \tfrac{250}{3}]\}}^{\triangle AGC}}{(600)(1000)}$$

$$= 491.3 \text{ mm}$$

7.4. Determine the moment of inertia of a rectangle about an axis through the centroid and parallel to the base.

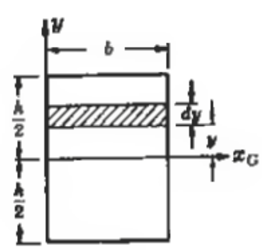


Fig. 7-9

Let us introduce the coordinate system shown in Fig. 7-9. The moment of inertia I_{x_G} about the x-axis passing through the centroid is given by $I_{x_G} = \int y^2\,da$. For convenience it is logical to select an element such that y is constant for all points in the element. The shaded area shown has this characteristic.

$$I_{x_G} = \int_{-h/2}^{h/2} y^2 b\,dy = b\left[\frac{y^3}{3}\right]_{-h/2}^{h/2} = \frac{1}{12}bh^3$$

This quantity has the dimension of a length to the fourth power, perhaps in^4 or m^4.

7.5. Derive the parallel-axis theorem for moments of inertia of a plane area.

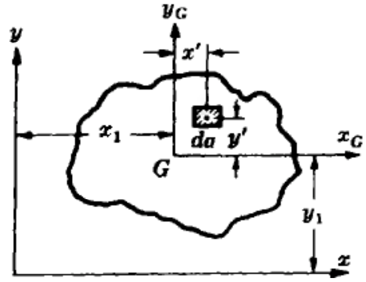


Fig. 7-10

Let us consider the plane area A shown in Fig. 7-10. The axes x_G and y_G pass through its centroid, whose location is presumed to be known. The axes x and y are located at known distances y_1 and x_1, respectively, from the axes through the centroid.

For the element of area da the moment of inertia about the x-axis is given by

$$dI_x = (y_1 + y')^2\,da$$

For the entire area A the moment of inertia about the x-axis is

$$I_x = \int dI_x = \int (y_1 + y')^2\,da = \int (y_1)^2\,da + 2\int y_1 y'\,da + \int (y')^2\,da$$

The first integral on the right is equal to $y_1^2 \int da = y_1^2 A$ because y_1 is a constant. The second integral on the right is equal to $2y_1 \int y'\,da = 2y_1(0) = 0$ because the axis from which y' is measured passes through the centroid of the area. The third integral on the right is equal to I_{x_G}, i.e., the moment of inertia of the area about the horizontal axis through the centroid. Thus

$$I_x = I_{x_G} + A(y_1)^2$$

A similar consideration in the other direction would show that

$$I_y = I_{y_G} + A(x_1)^2$$

This is the parallel-axis theorem for plane areas. It is to be noted that one of the axes involved in each equation must pass through the centroid of the area. In words, this may be stated as follows: The moment of inertia of an area with reference to an axis not through the centroid of the area is equal to the moment of inertia about a parallel axis through the centroid of the area plus the product of the same area and the square of the distance between the two axes.

The moment of inertia always has a positive value, with a minimum value for axes through the centroid of the area in question.

7.6. Find the moment of inertia of a rectangle about an axis coinciding with the base.

The coordinate system shown in Fig. 7-11 is convenient. By definition the moment of inertia about the x-axis is given by $I_x = \int y^2\,da$. For the element shown y is constant for all points in the element. Hence

$$I_x = \int_0^h y^2 b\,dy = b\left[\frac{y^3}{3}\right]_0^h = \frac{1}{3}bh^3$$

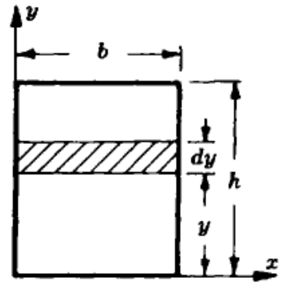


Fig. 7-11

This solution could also have been obtained by applying the parallel-axis theorem to the result obtained in Problem 7-4. This states that the moment of inertia about the base is equal to the moment of inertia about the horizontal axis through the centroid plus the product of the area and the square of the distance between these two axes. Thus

$$I_x = I_{x_G} + A(y_1)^2 = \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3$$

7.7. Determine the moment of inertia of a triangle about an axis coinciding with the base.

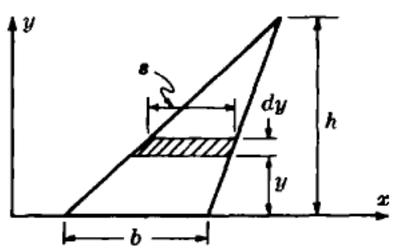


Fig. 7-12

Let us introduce the coordinate system shown in Fig. 7-12. The moment of inertia about the horizontal base is

$$I_x = \int y^2\,da$$

For the shaded element shown the quantity y is constant for all points in the element. Thus

$$I_x = \int_0^h y^2 s\,dy$$

By similar triangles, $s/b = (h-y)/h$, so that

$$I_x = \int_0^h y^2 \frac{b}{h}(h-y)\,dy = \frac{b}{h}\left[h\int_0^h y^2 dy - \int_0^h y^3 dy\right] = \frac{1}{12}bh^3$$

7.8. Determine the moment of inertia of a triangle about an axis through the centroid and parallel to the base.

Let the x_G-axis pass through the centroid and take the x-axis to coincide with the base as shown in Fig. 7-13.

From Fig. 7-3(b) the x_G-axis is located a distance of $h/3$ above the base. Also, the parallel-axis theorem tells us that

$$I_x = I_{x_G} + A(y_1)^2$$

But I_x was determined in Problem 7.7, and A and y_1 ($= h/3$) are known. Hence we may solve for the desired unknown, I_{x_G}. Substituting,

$$\frac{1}{12}bh^3 = I_{x_G} + \frac{1}{2}bh\left(\frac{h}{3}\right)^2 \qquad \text{or} \qquad I_{x_G} = \frac{1}{36}bh^3$$

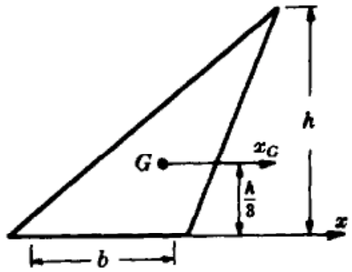


Fig. 7-13

7.9. Determine the moment of inertia of a circle about a diameter.

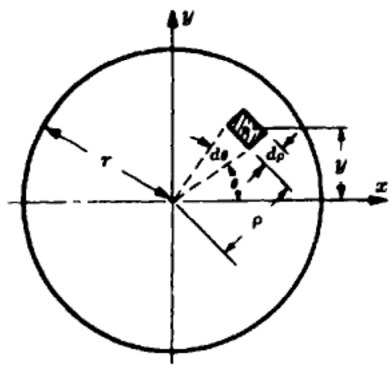


Fig. 7-14

Let us select the shaded element of area shown in Fig. 7-14, and work with the polar coordinate system. The radius of the circle is r.

To find I_x we have the definition $I_x = \int y^2\,da$.

But $y = \rho \sin\theta$ and $da = \rho\,d\theta\,d\rho$. Hence

$$I_x = \int_0^{2\pi}\int_0^r \rho^2 \sin^2\theta \rho\,d\theta\,d\rho = \int_0^{2\pi} \sin^2\theta\,d\theta\left[\frac{1}{4}\rho^4\right]_0^r$$

$$= \frac{r^4}{4}\int_0^{2\pi} \sin^2\theta\,d\theta = \frac{\pi r^4}{4}$$

If D denotes the diameter of the circle, then $D = 2r$ and $I_x = \pi D^4/64$. This is half the value of the polar moment of inertia of a solid circular area (see Problem 5.1).

The moment of inertia of a semicircular area about an axis coinciding with its base is

$$I_x = \frac{1}{2}\frac{\pi D^4}{64} = \frac{\pi D^4}{128}$$

7.10. Determine the moment of inertia about both the x- and y-axes as well as the corresponding radii of gyration of the plane area shown in Fig. 7-15.

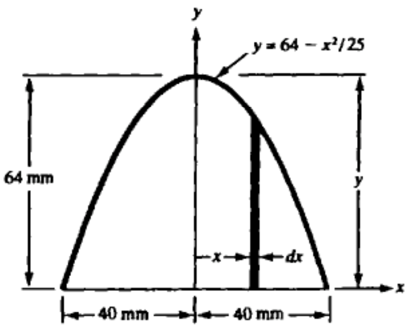


Fig. 7-15

Let us select the shaded element of width dx and altitude y shown in Fig. 7-15. From Problem 7.6 we have the moment of inertia of this element about the x-axis as

$$dI_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(dx)y^3$$

Now, we must integrate over all values of x from -40 mm to $+40$ mm to account for all such elements. Thus,

$$I_x = \int dI_x = \frac{1}{3}\int_{x=-40}^{x=40} y^3\,dx$$

$$= \frac{2}{3}\int_{x=0}^{x=40}\left[64 - \frac{x^2}{25}\right]^3 dx$$

$$= 3.197 \times 10^6\ \text{mm}^4$$

The same element may be employed to determine the moment of inertia of the entire area about the y-axis. By definition we have

$$dI_y = x^2\,da$$

which becomes

$$I_y = \int dI_y = \int_{x=-40}^{x=40} x^2 y\,dx$$

$$= 2\int_{x=0}^{x=40} x^2\left(64 - \frac{x^2}{25}\right)dx$$

$$= 1.092 \times 10^6\ \text{mm}^4$$

To determine the radii of gyration, it is first necessary to find the area under the curve. It is given by

$$A = \int y\,dx$$

$$= 2\int_{x=0}^{x=40}\left(64 - \frac{x^2}{25}\right)dx = 3413\ \text{mm}^2$$

from which we have

$$r_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{3.197 \times 10^6\ \text{mm}^4}{3413\ \text{mm}^2}} = 30.6\ \text{mm}$$

$$r_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{1.092 \times 10^6\ \text{mm}^4}{3413\ \text{mm}^2}} = 17.9\ \text{mm}$$

7.11. Two channel sections are attached to a cover plate 16 in long by $\frac{1}{2}$ in thick, as indicated in Fig. 7-16. Locate the centroid of the cross section and determine the moment of inertia and radius of gyration about an axis parallel to the x-axis and passing through the centroid.

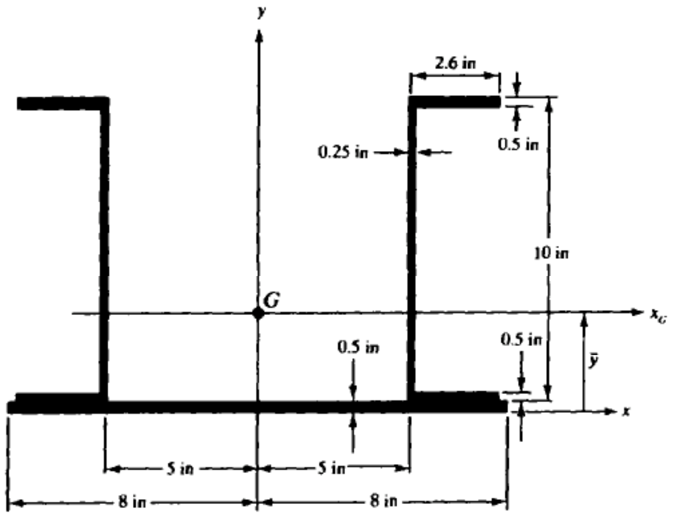


Fig. 7-16

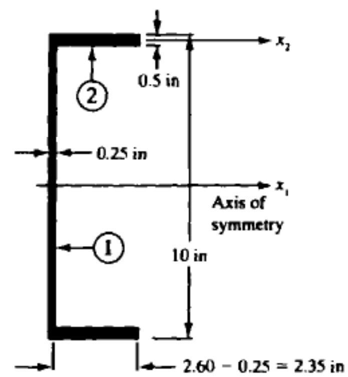


Fig. 7-17

Let us first consider a single channel section, as shown in Fig. 7-17. The area of the cross section is

$$A = 2(\tfrac{1}{2})(2.60 - 0.25) + 10(\tfrac{1}{4}) = 4.85\ \text{in}^2$$

and from Problem 7.4 together with the parallel-axis theorem we have the moment of inertia of the channel about an axis parallel to the x-axis and passing through the centroid of the channel (the x_1-axis) as

$$I_{ch} = \underbrace{\tfrac{1}{12}(\tfrac{1}{4})(10)^3}_{①} + 2\{\underbrace{\tfrac{1}{12}(2.35)(\tfrac{1}{2})^3}_{②} + \underbrace{(2.35)(\tfrac{1}{2})(5 - \tfrac{1}{4})^2}_{③}\}$$

$$= 73.90\ \text{in}^4$$

where term ① corresponds to the moment of inertia of the vertical rectangle about the x_1-axis, term ② corresponds to the moment of inertia of one horizontal rectangle about the x_2-axis through the centroid of the horizontal rectangle, and term ③ indicates the transfer term from the parallel axis theorem to pass from axis x_2 to axis x_1.

Now, we may write the moment of inertia of the entire assembly about the x-axis by applying the result of Problem 7.6 to the cover plate and applying the parallel axis theorem to I_{ch} to obtain

$$I_x = \tfrac{1}{3}(16)(\tfrac{1}{2})^3 + 2\{73.87 + 4.85(5.5)^2\} = 441.8\ \text{in}^4$$

The centroid of the cross section of the entire assembly is determined from the definition

$$\bar{y} = \frac{\Sigma\, y\, da}{A}$$

$$= \frac{\overbrace{(16)(\tfrac{1}{2})(\tfrac{1}{4})}^{③} + \overbrace{2[(4.85)(5.5)]}^{④}}{(16)(\tfrac{1}{2}) + 2[4.85]} = 3.13\ \text{in}$$

where the terms represented by ③ correspond to the horizontal cover plate and the terms numbered ④ correspond to the channels.

Now that we have located the centroidal axis x_G of the assembly, we may employ the parallel-axis theorem to transfer from the x- to the x_G-axis:

$$I_x = I_{x_G} + A(\bar{y})^2$$

$$441.8\ \text{in}^4 = I_{x_G} + (17.76\ \text{in}^2)(3.13\ \text{in})^2$$

$$I_{x_G} = 268.48\ \text{in}^4$$

The corresponding radius of gyration is

$$r_{x_G} = \sqrt{\frac{I_{x_G}}{A}} = \sqrt{\frac{268.48}{17.76}} = 3.89\ \text{in}$$

7.12. A plane section is in the form of an equilateral triangle, 200 mm on a side. From it is removed another equilateral triangle in such a manner that the width of the remaining section is 30 mm measured perpendicular to the sides of both equilateral triangles, as shown in Fig. 7-18. Determine the location of the centroid of the remaining (shaded) area as well as the moment of inertia about the axis through the centroid and parallel to the x-axis.

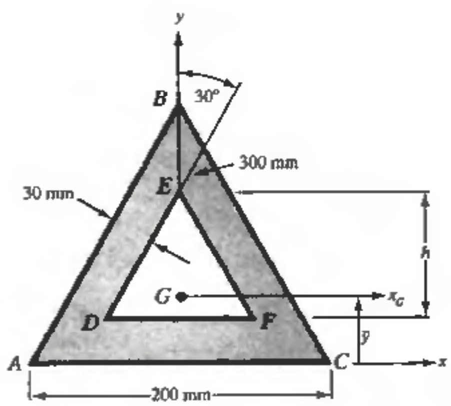


Fig. 7-18

It is necessary to determine the size of the inner triangle that has been removed. From the geometry of Fig. 7-18 it is evident that $BE = 60$ mm because of the 30° angle between BE and BC. Thus the altitude h of the "removed" triangle DEF is

$$h = 200 \cos 30 - 30 - 60 = 83.21\ \text{mm}$$

The length of a side of this triangle is

$$DF = \frac{83.21}{0.866} = 96.08\ \text{mm}$$

From symmetry the centroid lies on the y-axis and its location is found by the definition

$$\bar{y} = \frac{\int y\, da}{A} \qquad \text{or} \qquad \frac{\Sigma y\, dA}{A}$$

where the numerator represents the first moment of the area about the x-axis. Using the known location of the centroid of a triangle and its area, as given in the summary at the beginning of this chapter, we have

$$\bar{y} = \frac{\frac{1}{2}(200)(200\cos 30)(\frac{200}{3}\cos 30) - \frac{1}{2}(96.08)(83.21)\{30 + 83.21/3\}}{\frac{1}{2}(200)(200\cos 30) - \frac{1}{2}(96.08)(83.21)}$$

$$= 57.72\ \text{mm}$$

To determine the moment of inertia of the shaded area in Fig. 7-18, we begin by finding the moment of inertia of that area about the x-axis. This is accomplished by taking the moment of inertia of the outer triangle ABC about the x-axis using the result of Problem 7.7, then subtracting the moment of inertia of the inner triangle DEF about that same axis. This latter value is calculated by first determining the moment of inertia of DEF about an axis through the centroid of DEF using the result of Problem 7.8, then employing the parallel-axis theorem to transfer that value to the x-axis. Thus,

$$\begin{aligned} I_x &= \tfrac{1}{12}(200)\,(200\cos 30)^3 - \{\tfrac{1}{36}(96.08)\,(83.21)^3 + \tfrac{1}{2}(96.08)\,(83.21)\,[30 + 83.21/3]^2\} \\ &= 71.74 \times 10^6\ \text{mm}^4 \end{aligned}$$

Utilizing the parallel-axis theorem, we have

$$I_x = I_{x_G} + A(\bar{y})^2$$

$$71.74 \times 10^6\ \text{mm}^4 = I_{x_G} + \{\tfrac{1}{2}(200)\,(200\cos 30) - \tfrac{1}{2}(96.08)\,(83.21)\}\,(57.72\ \text{mm})^2$$

$$I_{x_G} = 27.35 \times 10^6\ \text{mm}^4$$

7.13. Determine the product of inertia of a rectangle with respect to the x- and y-axes indicated in Fig. 7-19.

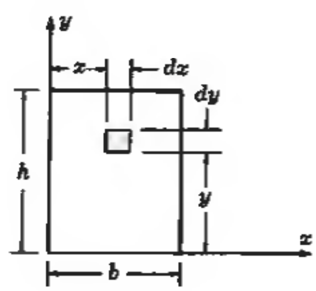


Fig. 7-19

We employ the definition $I_{xy} = \int xy\,da$ and consider the shaded element shown. Integrating,

$$\begin{aligned} I_{xy} &= \int_{y=0}^{y=h}\int_{x=0}^{x=b} xy\,dx\,dy = \int_{y=0}^{y=h}\left[\frac{x^2}{2}\right]_0^b y\,dy \\ &= \frac{b^2}{2}\left[\frac{y^2}{2}\right]_0^h = \frac{b^2h^2}{4} \end{aligned} \qquad (1)$$

7.14. Derive the parallel-axis theorem for product of inertia of a plane area.

In Fig. 7-20, the axes x_G and y_G pass through the centroid of the area A. The axes x and y are located the known distances y_1 and x_1, respectively, from the axes through the centroid.

For the element of area da the product of inertia with respect to the x- and y-axes is given by

$$dI_{xy} = (x_1 + x')\,(y_1 + y')\,dx\,dy$$

For the entire area the product of inertia with respect to the x- and y-axes becomes

$$\begin{aligned} I_{xy} &= \int dI_{xy} = \iint (x_1 + x')\,(y_1 + y')\,dx\,dy \\ &= \iint x_1y_1\,dx\,dy + \iint x'y_1\,dx\,dy + \iint x_1y'\,dx\,dy + \iint x'y'\,dx\,dy \end{aligned}$$

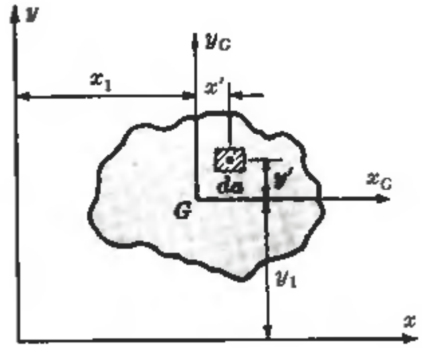


Fig. 7-20

The first integral on the right side equals $x_1 y_1 A$ since x_1 and y_1 are constants. The second and third integrals vanish because x' and y' are measured from the axes through the centroid of the area A. The fourth integral is equal to $I_{x_G y_G}$, that is, the product of inertia of the area with respect to axes through its centroid and parallel to the x- and y-axes. Thus, we have

$$I_{xy} = x_1 y_1 A + I_{x_G y_G} \tag{1}$$

This is the parallel-axis theorem for product of inertia of a plane area. It is to be noted that the x_G- and y_G-axes must pass through the centroid of the area. Also, x_1 and y_1 are positive only when the x- and y-coordinates have the location relative to the x_G-y_G system indicated in Fig. 7-20. Thus, care must be taken with regard to the algebraic signs of x_1 and y_1.

7.15. Determine I_{xy} for the angle section indicated in Fig. 7-21.

The area may be divided into the component rectangles as shown. For rectangle 1 we have, from (*1*) of Problem 7.13,

$$(I_{xy})_1 = \tfrac{1}{4}(10)^2(125)^2 = 39 \times 10^4 \text{ mm}^4$$

For rectangle 2 we employ (*1*) of Problem 7.14. The product of inertia of rectangle 2 about axes through its centroid and parallel to the x- and y-axes vanishes because these are axes of symmetry. Thus, for rectangle 2, $I_{x_G} = 0$. The parallel-axis theorem of Problem 7.14 thus becomes

$$(I_{xy})_2 = (42.5)(5)(65)(10) = 13.8 \times 10^4 \text{ mm}^4$$

For the entire angle section we thus have

$$I_{xy} = 39 \times 10^4 + 13.8 \times 10^4 = 52.8 \times 10^4 \text{ mm}^4$$

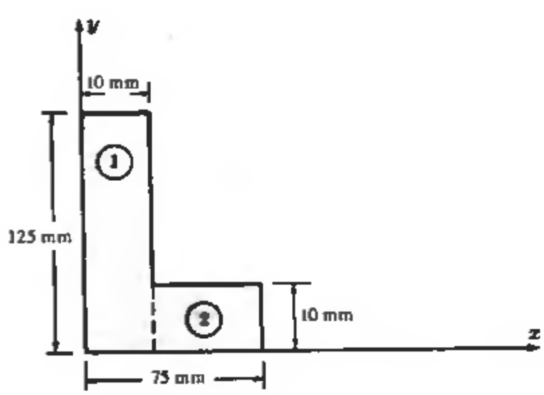


Fig. 7-21

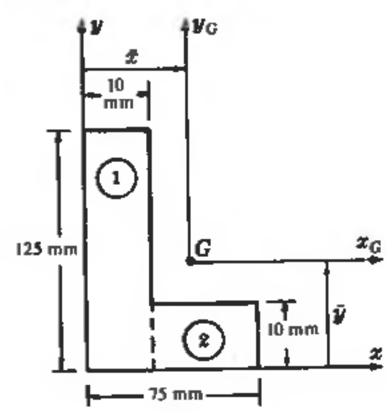


Fig. 7-22

7.16. Determine the product of inertia of the angle section of Problem 7.15 with respect to axes parallel to the x- and y-axes and passing through the centroid of the angle section. See Fig. 7-22.

It is first necessary to locate the centroid of the area, that is, we must find $\bar{x}$ and $\bar{y}$. We have

$$\bar{x} = \frac{125(10)(5) + 65(10)(42.5)}{125(10) + 65(10)} = 17.8 \text{ mm}$$

$$\bar{y} = \frac{125(10)(62.5) + 65(10)(5)}{125(10) + 65(10)} = 42.8 \text{ mm}$$

Now we employ the parallel-axis theorem of Problem 7.13; that is,

$$I_{xy} = x_1 y_1 A + I_{x_G y_G}$$

In Problem 7.15 we found $I_{xy} = 52.8 \times 10^4 \text{ mm}^4$. Thus

$$52.8 \times 10^4 = 17.8(42.8)(1900) + I_{x_G y_G}$$

whence

$$I_{x_G y_G} = -92 \times 10^4 \text{ mm}^4$$

7.17. Consider a plane area A and assume that I_x, I_y, and I_{xy} are known. Determine the moments of inertia I_{x_1} and I_{y_1} as well as the product of inertia $I_{x_1 y_1}$ for the set of orthogonal axes x_1-y_1 oriented as shown in Fig. 7-23. Determine also the maximum and minimum values of I_{x1}.

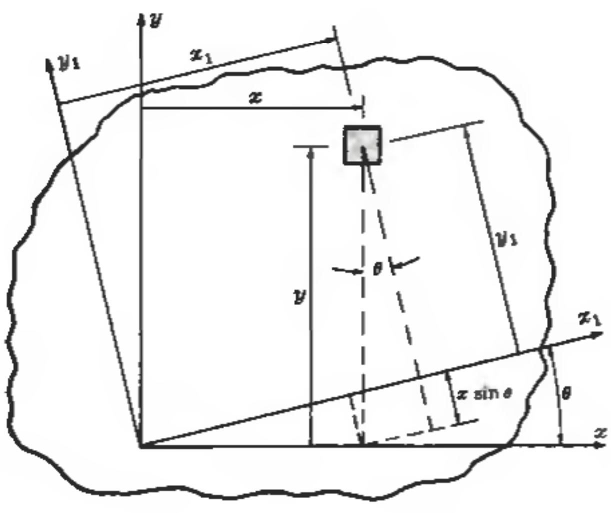


Fig. 7-23

The moment of inertia of the area with respect to the x_1-axis is

$$\begin{aligned} I_{x_1} &= \int y_1^2\,da = \int (y\cos\theta - x\sin\theta)^2\,da \\ &= \cos^2\theta \int y^2\,da + \sin^2\theta \int x^2\,da - 2\sin\theta\cos\theta \int xy\,da \\ &= I_x\cos^2\theta + I_y\sin^2\theta - 2I_{xy}\sin\theta\cos\theta \\ &= I_x\left(\frac{1+\cos 2\theta}{2}\right) + I_y\left(\frac{1-\cos 2\theta}{2}\right) - I_{xy}\sin 2\theta \end{aligned}$$

Or

$$I_{x_1} = \left(\frac{I_x + I_y}{2}\right) + \left(\frac{I_x - I_y}{2}\right)\cos 2\theta - I_{xy}\sin 2\theta \tag{1}$$

Analogously, I_{y_1} may be obtained from (*1*) by replacing θ by $\theta + \pi/2$ to yield

$$I_{y_1} = \left(\frac{I_x + I_y}{2}\right) - \left(\frac{I_x - I_y}{2}\right)\cos 2\theta + I_{xy}\sin 2\theta \tag{2}$$

The value of θ that renders I_{x_1} maximum or minimum is found by setting the derivative of Eq. (*1*) with respect to θ equal to zero. Thus, since I_x, I_y, and I_{xy} are constants we have from (*1*)

$$\frac{dI_{x_1}}{d\theta} = -(I_x - I_y)\sin 2\theta - 2I_{xy}\cos 2\theta = 0$$

Solving,

$$\tan 2\theta = -\frac{I_{xy}}{\left(\frac{I_x - I_y}{2}\right)} \tag{3}$$

Fig. 7-24

Equation (*3*) has the convenient graphical interpretation shown in Cases I and II of Fig. 7-24.

If now the values of 2θ given by (*3*) are substituted into (*1*), we obtain

$$(I_{x_1})_{\substack{\max \\ \min}} = \left(\frac{I_x + I_y}{2}\right) \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2} \tag{4}$$

where the positive sign refers to Case I and the negative sign to Case II. These maximum and minimum values of moment of inertia correspond to axes defined by (*3*). The maximum and minimum values of moment of inertia are termed *principal moments of inertia* and the corresponding axes are termed *principal axes.*

We may now determine $I_{x_1y_1}$ from

$$\begin{aligned} I_{x_1y_1} &= \int x_1 y_1\, da \\ &= \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\, da \\ &= \cos^2\theta \int xy\, da - \sin^2\theta \int xy\, da \\ &\quad + \sin\theta\cos\theta \int y^2\, da - \sin\theta\cos\theta \int x^2\, da \\ &= I_{xy}(\cos^2\theta - \sin^2\theta) + (I_x - I_y)\sin\theta\cos\theta \\ &= \left(\frac{I_x - I_y}{2}\right)\sin 2\theta + I_{xy}\cos 2\theta \end{aligned} \tag{5}$$

From (5), $I_{x_1y_1}$ vanishes if

$$\tan 2\theta = -\frac{I_{xy}}{\left(\dfrac{I_x - I_y}{2}\right)}$$

which is identical to condition (3). Since (3) defined principal axes, it follows that the product of inertia vanishes for principal axes.

7.18. **A structural aluminum 6 Z 5.42 section has the nominal dimensions indicated in Fig. 7-25. Determine I_x, I_y, I_{xy} and also the maximum and minimum values of the moment of inertia with respect to axes through the point O.**

The section may be divided into the component rectangles ①, ②, and ③ as indicated. The result obtained in Problem 7.4, together with the parallel-axis theorem given in Problem 7.5, may be used to determine I_x and I_y:

$$I_x = \tfrac{1}{12}(\tfrac{3}{8})(6)^3 + 2[\tfrac{1}{12}(3\tfrac{1}{8})(\tfrac{3}{8})^3 + (3\tfrac{1}{8})(\tfrac{3}{8})(2\tfrac{13}{16})^2] = 25.27\ \text{in}^4$$

$$I_y = \tfrac{1}{12}(6)(\tfrac{3}{8})^3 + 2[\tfrac{1}{12}(\tfrac{3}{8})(3\tfrac{1}{8})^3 + (\tfrac{3}{8})(3\tfrac{1}{8})(1\tfrac{3}{4})^2] = 9.08\ \text{in}^4$$

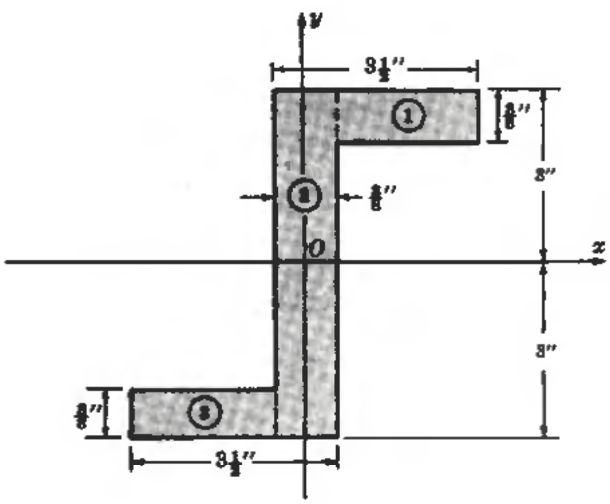


Fig. 7-25

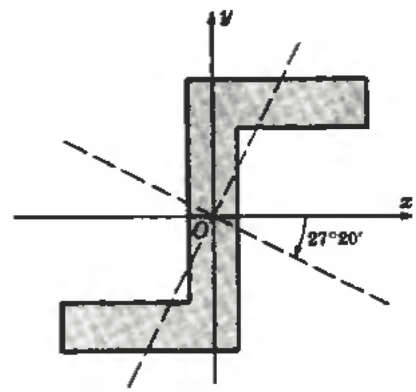


Fig. 7-26

The product of inertia with respect to the x- and y-axes may be determined through use of the parallel-axis theorem for product of inertia as given in Problem 7.14. It is to be noted that the product of inertia of each of the component rectangles about axes through the centroid of each component and parallel to the x- and y-axes vanishes because these are axes of symmetry. Hence, from (1) of Problem 7.14 we have for the entire Z-section

$$I_{xy} = 2[(\tfrac{7}{4})(2\tfrac{13}{16})(3\tfrac{1}{8})(\tfrac{3}{8})] = 11.6\ \text{in}^4$$

The maximum and minimum values of moment of inertia with respect to axes through the point O may be found from (4) of Problem 7.17. From that equation

$$(I_{x_1})_{\substack{\max\\ \min}} = \left(\frac{I_x + I_y}{2}\right) \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2}$$

$$= \left(\frac{25.27 + 9.08}{2}\right) \pm \sqrt{\left(\frac{25.27 - 9.08}{2}\right)^2 + (11.6)^2}$$

$$(I_{x_1})_{\max} = 31.38\ \text{in}^4 \qquad (1)$$

$$(I_{x_1})_{\min} = 2.98\ \text{in}^4 \qquad (2)$$

The orientation of these principal moments of inertia is found from (3) of Problem 7.17 to be

$$\tan 2\theta = -\frac{I_{xy}}{\left(\dfrac{I_x - I_y}{2}\right)}$$

$$= -\frac{11.6}{\left(\dfrac{25.27 - 9.08}{2}\right)}$$

$$\theta = -27°20', \qquad -117°20' \tag{3}$$

The principal moments of inertia given in (1) and (2) correspond to the principal axes given by (3). These principal axes are represented by the dashed lines in Fig. 7-26.

Supplementary Problems

7.19. The structural channel section has welded to it a horizontal reinforcing plate as shown in cross section in Fig. 7-27. Determine the y-coordinate of the centroid of the composite section. *Ans.* $\bar{y} = 4.56$ in

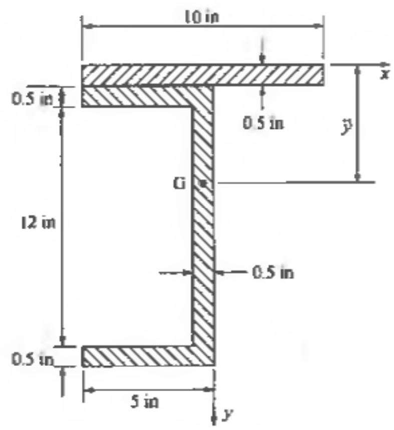


Fig. 7-27

7.20. The shaded area shown in Fig. 7-28 is bounded by a circular arc and a chord. Determine the location of the centroid of the area with respect to the center of the circular arc.

Ans. $\bar{y} = \dfrac{4R}{3} \cdot \dfrac{(\sin^3 \theta)}{(2\theta - \sin 2\theta)}$

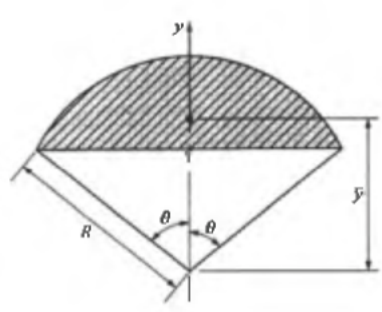


Fig. 7-28

7.21. An area consists of a circle of radius R from which a rectangle of dimensions $a \times 3a$ has been removed, as shown in Fig. 7-29. Determine the moment of inertia of the shaded area about the x- and also the y-axes.

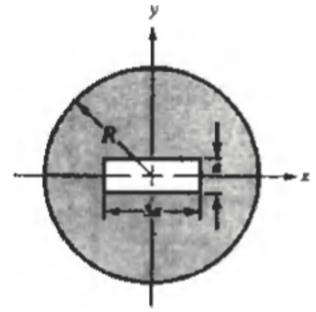


Fig. 7-29

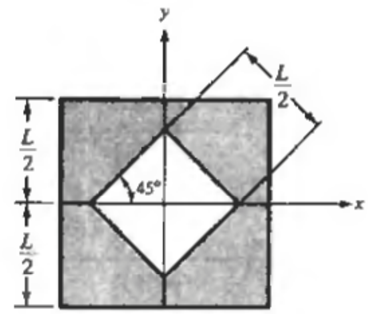


Fig. 7-30

Ans. $I_x = \dfrac{\pi R^4}{4} - \dfrac{a^4}{4}$, $\quad I_y = \dfrac{\pi R^4}{4} - \dfrac{9a^4}{4}$

7.22. The shaded area in Fig. 7-30 results from removing the central square from the outer square. Determine the moment of inertia of the net area about the x-axis. *Ans.* $I_x = 0.0781L^4$

7.23. A thin rectangular sheet has semicircular and also triangular areas removed, as shown in Fig. 7-31. Locate the centroid of the sheet and determine the moment of inertia about the horizontal axis passing through the centroid. *Ans.* $\bar{y} = 370.8$ mm, $I_{x_G} = 9937 \times 10^6$ mm^4

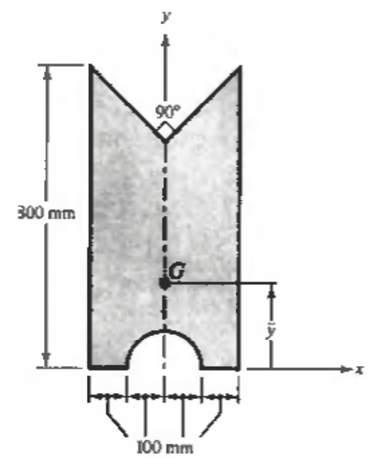


Fig. 7-31

7.24. A trapezoidal area has the dimensions indicated in Fig. 7-32. Determine the location of the centroid as well as the moment of inertia about an axis through the centroid and parallel to the x-axis.
Ans. $\bar{y} = 44.4$ mm. $I_{x_G} = 24.14 \times 10^6$ mm^4

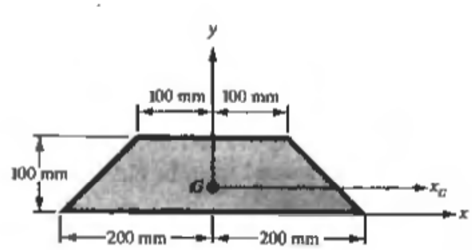


Fig. 7-32

7.25. A thin-walled section ($t \ll a$) has the configuration indicated in Fig. 7-33. Locate the centroid of the cross section and determine the moment of inertia of the area about an axis passing through the centroid and parallel to the x-axis. *Ans.* $\bar{y} = a$, $I_{x_G} = 5.33a^3t + at^3/6$

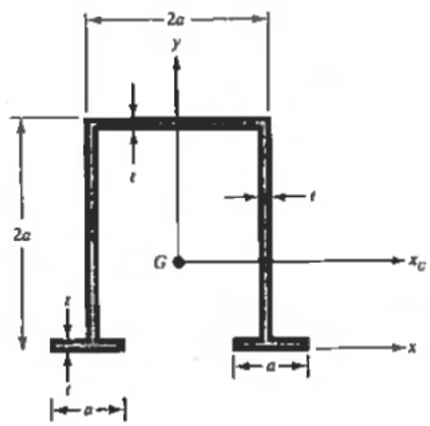


Fig. 7-33

7.26. An area of circular cross section from which three circular holes have been removed is shown in Fig. 7-34. Determine the location of the centroid of the section and the moment of inertia of an axis passing through the centroid and parallel to the x-axis. *Ans.* $\bar{y} = -R/10$, $I_{x_G} = 0.737R^4$

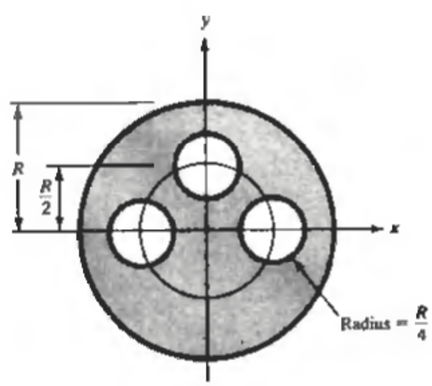


Fig. 7-34

7.27. Determine the moment of inertia of the diamond-shaped figure shown in Fig. 7-35 with respect to the horizontal axis of symmetry. *Ans.* $I_{x_G} = 85.4 \text{ in}^4$

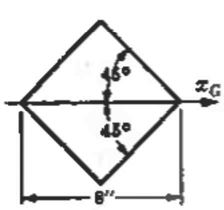


Fig. 7-35

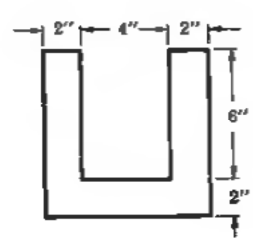


Fig. 7-36

7.28. Determine the moment of inertia of a channel-type section about a horizontal axis through the centroid. Refer to Fig. 7-36. What is the radius of gyration about this same axis?
Ans. $I_{x_G} = 231\ \text{in}^4$, $r_{x_G} = 2.40\ \text{in}$

7.29. Locate the centroid of the channel-type section shown in Fig. 7-37 and determine the moment of inertia of the cross-sectional area about a horizontal axis through the centroid.
Ans. $\bar{y} = 38.33\ \text{mm}$, $I_{x_G} = 33 \times 10^6\ \text{mm}^4$

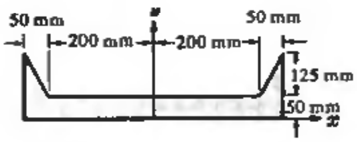


Fig. 7-37

7.30. A plane area has the shape of a parallelogram as shown in Fig. 7-38. The y- and z-axes pass through the centroid of the area. Determine I_y and I_z. *Ans.* $I_y = \frac{1}{12}bh^3$, $I_z = \frac{1}{12}hb(b^2 + c^2)$

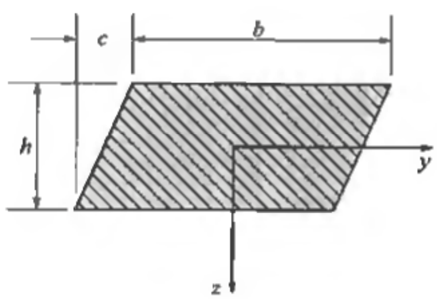


Fig. 7-38

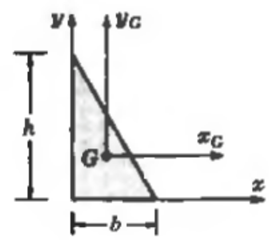


Fig. 7-39

7.31. Determine the product of inertia of a triangle with respect to the x- and y-axes indicated in Fig. 7-39.
Ans. $b^2h^2/24$

7.32. Determine the product of inertia of the triangle shown in Fig. 7-39 with respect to the axes x_G and y_G passing through the centroid. *Ans.* $-b^2h^2/72$

7.33. For the plane area in Fig. 7-40 determine the moments of inertia and product of inertia with respect to the x_G- and y_G-axes passing through the centroid. Also, determine the principal second moments of area with respect to the centroid.
Ans. $I_{x_G} = 400 \times 10^6\ \text{mm}^4$; $I_{y_G} = 147 \times 10^6\ \text{mm}^4$; $I_{x_Gy_G} = -58 \times 10^6\ \text{mm}^4$; $(I_{x_1})_{max} = 805 \times 10^6\ \text{mm}^4$; $(I_{x_1})_{min} = 142 \times 10^6\ \text{mm}^4$

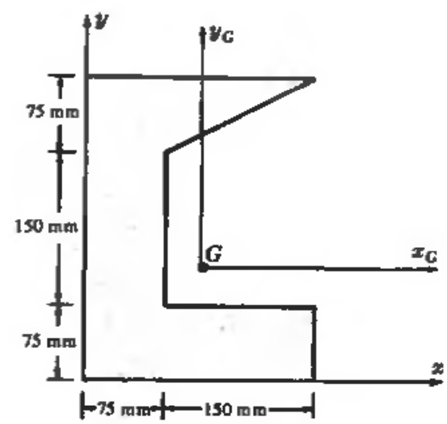


Fig. 7-40